Geometric constraint algorithm for field theories with boundaries: a different Hamiltonian point of view.

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CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

Motivation

- Field theories with **boundaries** are important in physics (free boundary problems in elastic media, fluids, solids, topological insulators,...).
- Boundaries in **gravitational physics**:
  - Asymptotic infinity.
  - Horizons as inner boundaries: the membrane paradigm.
  - Holography.
  - Isolated horizons and black holes in LQG.
- **Hamiltonian formulation of field theories with boundaries:**
  - Identification of physical degrees of freedom.
  - Is it possible/necessary to make a distinction between “bulk d.o.f” and “boundary d.o.f.”?
  - This is a very relevant question in gravitational physics (holography?)
  - Black hole entropy and boundary d.o.f. Are we really counting (classical or quantum) microstates "living on the horizon"?
  - It must be relevant for quantization. If a natural split between "bulk d.o.f." and "boundary d.o.f." exists then it is natural to postulate $\mathcal{H} = \mathcal{H}_{\text{bulk}} \otimes \mathcal{H}_{\text{boundary}}$ and study them separately.

Motivation

- **Boundary conditions and d.o.f.**
  - Boundary conditions for field theories restrict the behavior in the bulk, but not completely.
  - Are values at the boundary determined from those on the bulk ("by continuity")? Should not it be the other way around?
  - What is a degree of freedom? Not so easy if you think a little bit about it, even for finite mechanical systems...($\mathbb{S}^1 \times \mathbb{S}^1$ different from $\mathbb{S}^2$, right?)
  - It is easier to count d.o.f. than to define what they are... (dimension of the configuration space)
  - Should we care? We usually act as if we actually could live without a precise knowledge of what d.o.f. mean in field theories but then it is not surprising at all that we have run into problems at some point.
  - Does it make sense to talk about purely quantum d.o.f. before understanding the classical d.o.f. (isolated horizons)?
- How do you treat them when quantizing?

Should boundary conditions be interpreted as constraints?

Summary (of the rest of the talk, that is...)

1. Hamiltonian formulation for field theories with boundaries.
2. Simple models: a scalar field with boundary conditions.
3. The geometric constraint algorithm of Gotay-Nester-Hinds.
4. Scalar fields:
   - Dirichlet boundary conditions.
   - Robin boundary conditions.
5. Electromagnetic field:
   - Perfect conductor boundary conditions.
   - Neumann boundary conditions.
6. Conclusions and comments.

A very simple model: a **scalar field** defined on the segment $[0,1]$ with **Dirichlet boundary conditions**

$$S[\varphi,\psi_0,\psi_1] = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{1}{2}\int_0^1 \mathrm{d}x(\dot{\varphi}^2 - \varphi'^2) + \psi_0\big(\varphi(0)-\varphi_0\big) + \psi_1\big(\varphi(1)-\varphi_1\big)\right)$$

- The configuration variables are $\varphi(x)$, ψ_0 and ψ_1.
- ψ_0 and ψ_1 are Lagrange multipliers that enforce the boundary conditions $\varphi(0)=\varphi_0$ and $\varphi(1)=\varphi_1$.
- $\varphi_0\,,\varphi_1 \in \mathbb{R}$, (boundary values of φ).
- $\varphi \in C^2(0,1) \cap C^1[0,1]$ (we need second order derivatives).

Do we get the right field equations?

We should better check...

Field equations: variations of the action

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \mathrm{d}t \left(\int_0^1 \mathrm{d}x(-\ddot{\varphi}(x) + \varphi''(x))\delta\varphi(x) - \varphi'(x)\delta\varphi(x)\Big|_0^1 \right) \\ &+ \int_{t_1}^{t_2} \mathrm{d}t(\varphi(0) - \varphi_0)\delta\psi_0 + \int_{t_1}^{t_2} \mathrm{d}t(\varphi(1) - \varphi_1)\delta\psi_1 \\ &+ \int_{t_1}^{t_2} \mathrm{d}t\psi_0\delta\varphi(0) + \int_{t_1}^{t_2} \mathrm{d}t\psi_1\delta\varphi(1)\end{aligned}$$

$$\begin{aligned}&\ddot{\varphi}(x) - \varphi''(x) = 0\,, \quad x \in (0,1) \\ &\varphi(0) = \varphi_0 \\ &\varphi(1) = \varphi_1 \qquad \checkmark \\ &\psi_1 - \varphi'(1) = 0 \\ &\psi_0 + \varphi'(0) = 0\end{aligned}$$

A crash course on the Hamiltonian treatment of constrained systems: the Dirac algorithm (circa 1950).

1. Take **Lagrangian** L.
2. Compute **canonical momenta** $p_i = \frac{\partial L}{\partial \dot{q}_k}$.
3. If not all the $\dot{q}_k$ can be solved in terms of the $p_k \rightsquigarrow$ **primary constraints** $\phi_j(q, p) = 0$.
4. **Total Hamiltonian** $H_T = H + u_j \phi_j$, where $H = p_k \dot{q}_k - L$.
5. Stability of the constraints in time evolution $\{\phi_j, H_T\} = 0$ may be the source of more (secondary) constraints or restrict the values of the u_j.
6. At some point nothing new comes out of the stability conditions. The algorithm stops.
7. Classification of constraints as **first class** (commute with all the constraints) or **second class** (otherwise).
8. Second class constraints are bad. Get rid of them either by "solving" them or introducing the Dirac bracket.

Back to our example..

- **Canonical momenta**:

$$\pi(x) := \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)} = \dot{\varphi}(x)\,,\, p_0 := \frac{\partial \mathcal{L}}{\partial \psi_0} = 0\,,\, p_1 := \frac{\partial \mathcal{L}}{\partial \psi_1} = 0$$

- **Primary constraints** $p_0 = 0$ and $p_1 = 0$.
- Non-zero **Poisson brackets**

$$\{\varphi(x), \pi(y)\} = \delta(x,y)\,,\, \{\psi_0, p_0\} = 1\,,\, \{\psi_1, p_1\} = 1$$

- **Total hamiltonian**

$$H_T = -\psi_0(\varphi(0) - \varphi_0) - \psi_0(\varphi(0) - \varphi_0) + \xi_0 p_0 + \xi_1 p_1 + \frac{1}{2}\int_0^1 \mathrm{d}x \left(\pi^2 + \varphi'^2\right).$$

- Here ξ_0 and ξ_1 are the “Lagrange multipliers” that “enforce the primary constraints” in the Dirac algorithm.

Before going further just a little question...

What is the value of $\{\varphi(0), \pi(0)\}$?, Is it 1?, Is it $\delta(0,0)$?

This is not an academic question

Secondary constraints (at $x = 0$, analogously at $x = 1$)

$$\{H_T, p_0\} = \varphi_0 - \varphi(0) = 0 \quad \text{(OK)}$$

$$\{H_T, \varphi_0 - \varphi(0)\} = -\int_0^1 \mathrm{d}x \pi(x) \underbrace{\{\pi(x), \varphi(0)\}}_{-\delta(x,0)} = \pi(0) = 0 \quad \text{(uhm...)}$$

$$\begin{aligned}\{H_T, \pi(0)\} &= -\{\varphi(0), \pi(0)\}\psi_0 + \int_0^1 \mathrm{d}x \varphi'(x)\{\varphi'(x), \pi(0)\} \\ &= -\{\varphi(0), \pi(0)\}\psi_0 + \varphi'(x)\{\varphi(x), \pi(0)\}\Big|_0^1 \\ &\quad - \int_0^1 \mathrm{d}x \varphi''(x)\{\varphi(x), \pi(0)\} \quad \text{(???)}\end{aligned}$$

The algorithm **crashes badly**. One has to be careful...

LAGRANGIAN DYNAMICS (FINITE #OF D.O.F).

- Introduce a **configuration space** Q. A finite dimensional differential manifold. (for example: $\mathbb{S}^1$ for the planar pendulum).
- Define a **Lagrangian** $L : TQ \to \mathbb{R}$.
- Fix **initial and final** configurations $Q_1, Q_2 \in Q$ at t_1 and t_2, $(t_1 < t_2)$.
- Introduce a space of **curves on** Q. A common choice is $\mathcal{C}(Q_1, Q_2, [t_1, t_2]) := \{\Phi \in C^2([t_1, t_2], \Omega) | \Phi(t_1) = Q_1, \Phi(t_2) = Q_2\}$ (this has the structure of a Banach manifold).
- Define the action $\mathcal{S} : \mathcal{C}(Q_1, Q_2, [t_1, t_2]) \to \mathbb{R}$

$$S(\Phi) := \int_{t_1}^{t_2} L(\Phi(t), \dot{\Phi}(t)) \mathrm{d}t \,.$$

- Find the stationary points of S. These are given as solutions to differential equations (**equations of motion**).
- This is a **variational approach** (Hamilton's principle).

- It is defined on the **cotangent bundle** T^*Q of the configuration space (phase space).
- The phase space carries a canonical, (strongly) non-degenerate, symplectic form $\Omega \in \Lambda^2(T^*Q)$
  - **Canonical:** defined by the bundle structure (projections,...).
  - **Strongly non-degenerate:** The map $\flat : T(T^*Q) \to T^*(T^*Q) : X \mapsto \flat(X) = i_X\Omega$ is an isomorphism.
- **From the Lagrangian to the Hamiltonian**: A Lagrangian L defines a fiber derivative $FL : TQ \to T^*Q : w \mapsto FL(w)$ through

$$\langle v, FL(w)\rangle := \left.\frac{d}{dt}\right|_{s=0} L(w+sv)\,.$$

(here $\langle\cdot|\cdot\rangle$ is the natural pairing between elements of TQ and T^*Q over the same base point).

- The **Hamiltonian** is defined by $H \circ FL(w) = \langle w|FL(w)\rangle - L(w)\,,$ $w \in TQ$. The fiber derivative defines the **canonical momenta** too.

Hamiltonian dynamics (finite #of d.o.f).

- For regular (hyperregular) systems the fiber derivative is a local (global) **diffeomorphism**. Otherwise the system is **singular**.
- For hyperregular systems the Hamiltonian is defined on the full T^*Q, otherwise it is defined only on $FL(TQ)$.
- **Two steps** to get the dynamics of the system:
  - Obtain a Hamiltonian vector field $X \in \mathfrak{X}(T^*Q)$ from the Hamiltonian H by solving the equation $i_X\Omega = dH$.
  - Find the integral curves of X to get the time evolution of the system (by projection onto Q).
- For hyperregular systems the equation $i_X\Omega = dH$ is rather trivial because H is defined on the full T^*Q and $\flat$ is invertible. In fact, $X = \flat^{-1}(\mathrm{d}H)$. In canonical coordinates:

$$X = \left(\frac{\partial H}{\partial p_k}, -\frac{\partial H}{\partial q_k}\right).$$

- The standard Hamilton equations are obtained through these two steps.

Singular Hamilt. systems (finite #of d.o.f).

- The domain of the Hamiltonian ($\mathcal{M} := FL(TQ)$) is a subset of T^*Q (usually an embedded submanifold). This is called the **primary constraint** manifold.
- **Problem:** How do I make sense of all this (for instance, $i_X\Omega = dH$)?
- Two possible attitudes:

  1. Extend somehow the Hamitonian to the full phase space T^*Q.
  2. Restrict oneself to work on the primary constraint manifold $\mathcal{M}$.

- The first idea is the basis of the **Dirac** approach to the Hamiltonian description of singular systems. This is ultimately justified because it provides a reasonably simple approach to quantization (Dirac's book is titled *Lectures on Quantum Mechanics*...)
- The second is the starting point of the so called **Gotay-Nester-Hinds** (GNH) method (1978).

The Dirac algorithm revisited.

1. Identify the **primary constraint** submanifold $\mathcal{M}$ and describe it as $\Phi_k = 0$ in a particular canonical coordinate system in phase space. These functions are known as the **primary constraints**.
2. Find the **Hamiltonian** H on $\mathcal{M}$.
3. Introduce a "large enough family" of extensions of H to the full phase space T^*Q (the "total Hamiltonian" H_T). This is done in practice by taking one particular extension of H and adding to it a linear combination of the Φ_k multiplied by arbitrary (at this stage) functions μ_k in T^*Q (why Lagrange multipliers?). $H_T = H + \mu_k \Phi_k$.
4. Obtain the corresponding family of vector fields X_μ by solving the equation $i_X \Omega = dH$ (no problem here if Ω strongly non-degenerate).
5. Require that the X_μ are tangent to $\mathcal{M}$. In order to accomplish this it may be necessary to (non-exclusively):
   - Restrict oneself to a submanifold of $\mathcal{M}$ (defined by **secondary constraints**)
   - Restrict the functional form of the functions μ_k.

The Dirac algorithm.

6. Repeat as many times as necessary if X is not tangent to the submanifold $\mathcal{M}_k$ obtained at the step k.
7. The final outcome is a certain submanifold $\mathcal{N}$ of $\mathcal{M}$, and a concrete functional form for the functions μ_k that **may** or **may not** involve **arbitrary parameters**.
8. $\mathcal{N}$ is usually described as the null set of certain of functions ("constraints") but has an intrinsic meaning (it is a submanifold of $\mathcal{M}$).
9. When these μ_k are plugged into the expression of the total Hamiltonian H_T we get a Hamiltonian H' that defines a consistent dynamics in the sense that the Hamiltonian vector field X solves $i_X\Omega = \mathrm{d}H'$ and is tangent to $\mathcal{N}$.
10. The **integral curves** of X give the **time evolution** of the system.

6. $\mathcal{N}$ can be classified according to its **geometric properties** by defining

$$T\mathcal{N}^{\perp} := \{Z \in T\mathcal{M}|_{\mathcal{N}} : \Omega|_{\mathcal{N}}(Z, X) = 0, \forall X \in \underline{T\mathcal{N}}\}\,, \underline{T\mathcal{N}} := j_*(T\mathcal{N})\,.$$

- **First class:** $TN^{\perp} \subset \underline{T\mathcal{N}}$
- **Second class:** $TN^{\perp} \cap \underline{T\mathcal{N}} = \{0\}$
- **Isotropic:** $\underline{T\mathcal{N}} \subset TN^{\perp}$
- **Lagrangian:** $\underline{T\mathcal{N}} = TN^{\perp}$
- **Mixed:** The rest of them.

7. This generalizes in a neat geometric way the original classification introduced by Dirac in terms of constraint functions. This classification relies on the properties of the (pre)symplectic form induced by Ω on $\mathcal{N}$.
8. **Dirac bracket**: For mechanical systems (finite #d.o.f) there exists a submanifold $\mathcal{P}$ of $\mathcal{M}$ such that: $\mathcal{N}$ is a first class submanifold of $(\mathcal{P}, p^*\Omega)$ (Śniatycki theorem). The Dirac bracket is the Poisson bracket defined by $p^*\Omega$. Here p denotes the imbedding $p : \mathcal{P} \to \mathcal{M}$.

The GNH algorithm for finite-dim. systems

1. Find the **primary constraint submanifold** $\mathcal{M}_1 := FL(TQ) \subset T^*Q$.
2. Compute the pullback ω of the canonical symplectic form Ω to $\mathcal{M}$ and the (uniquely defined) Hamiltonian H on $\mathcal{M}$.
3. The goal of the GNH algorithm is to find the *maximal* submanifold $\mathcal{N} \subset \mathcal{M}$ where the equation

$$(i_X\omega - \mathrm{d}H)|_{\mathcal{N}} = 0 \tag{1}$$

   can be solved for X and gives rise to vector fields $X : \mathcal{N} \to T\mathcal{N}$, i.e. **tangent to** $\mathcal{N}$. This is done in successive steps:
4. For $m \in \mathcal{M}_1$ define the map $\flat : T_m\mathcal{M}_1 \to T_m^*\mathcal{M} : X \mapsto i_X\omega$.
5. Find the set $\mathcal{M}_2 := \{m \in \mathcal{M}_1 : \mathrm{d}H(m) \in \flat(T_m\mathcal{M}_1)\} \subset \mathcal{M}_1$.
6. Find $X : \mathcal{M}_2 \to T\mathcal{M}_1$ solving $(i_X\omega - \mathrm{d}H)|_{\mathcal{M}_2} = 0$. These vector fields are tangent to $\mathcal{M}_1$ by construction but not necessarily to $\mathcal{M}_2$. If this happens we iterate the procedure.

The GNH algorithm for finite-dim. systems

7. Find $\mathcal{M}_3 := \{m \in \mathcal{M}_2 : \mathrm{d}H(m) \in \flat(T_m\mathcal{M}_2)\} \subset \mathcal{M}_2 \subset \mathcal{M}_1$, solve $(i_X\omega - \mathrm{d}H)|_{\mathcal{M}_3} = 0$...
8. By doing this we obtain the sets

$$\mathcal{M}_{k+1} := \{m \in \mathcal{M}_k : \mathrm{d}H(m) \in \flat(T_m\mathcal{M}_k)\} \subset \mathcal{M}_k \subset \mathcal{M}_{k-1} \subset \cdots \subset \mathcal{M}_1\,,$$

   and the solution to

$$(i_X\omega - \mathrm{d}H)|_{\mathcal{M}_k} = 0$$

9. If the smallest $n \geq 1$ such that $\mathcal{M}_{n+1} = \mathcal{M}_n$ exists, the manifold $\mathcal{N} := \mathcal{M}_n$ and the (generically non-unique) vector fields $X : \mathcal{N} \to T\mathcal{N}$ constitute the Hamiltonian description of the system.
10. $\mathcal{N}$ is built as an embedded submanifold of $\mathcal{M} \subset T^*Q$ and X is tangent to it.
11. The integral curves of X give the dynamics of the system.

According to Dirac:

> *...It is then merely a formal matter to pass from this finite number of degrees of freedom to the infinite number of degrees of freedom which we need for a field theory.*
>
> P.A.M Dirac, *Lectures on quantum Mechanics*.

But we have seen that even for very simple field theories some nagging problems may show up. The generalization to field theories is not really a "formal matter". **One has to be very careful.**

A list of problems:

1. **Domain problems:** The **natural domains** for the Lagrangians that describe the field theories that we are interested in are not that simple:
   - They are not tangent bundles (though they are dense generalized submanifolds of a tangent bundle of the form $T_D Q$). The *fields* and the *velocities* do not live in the same spaces.
   - They are non-trivial functional spaces (Sobolev spaces).

2. The **configuration spaces** for field theories are **infinite dimensional manifolds**. Even the tamest of these spaces are way richer than the finite dimensional manifolds of classical mechanics.

    - **Finite dimensional configuration** spaces are finite dimensional real differential manifolds. They are hence modeled on finite dimensional euclidean spaces with a very simple metric topology (all the possible norms are **equivalent**).
    - The simplest **infinite dimensional configuration spaces** are Banach (in many cases Hilbert) manifolds, i.e. they are modeled on Banach spaces.
    - Infinite dimensional Banach spaces are very rich (**non-equivalent norms**, many possible metric topologies...) hence one has to carefully make many choices in this regard.

3. **Problems with the standard algorithms**: The naive generalizations of the Dirac or GNH algorithms to the infinite dimensional case, even for the simplest of the situations (scalar fields in $\mathbb{R}^n$) present some very unpleasant features. The worst one is that **they never stop** so the description of the dynamical manifold $\mathcal{N}$ ma be complicated because one has to deal with an infinite number of constraints.
4. This situation usually requires to trade **Banach manifolds** for **Fréchet manifolds** which are harder to handle.
5. **Problems with the symplectic form:** In many cases one has to deal with symplectic forms that are, at best, only weakly non-degenerate so the invertibility of the $\flat$ maps, that trivializes many of the computations for finite-dimensional mechanical systems, cannot be taken for granted.
6. There is **no infinite-dimensional generalization** of Śnyatycki's theorem, hence, one may run into serious difficulties if one insists in using Dirac's quantization. Reduced space quantization may be unavoidable if this happens (of course if you can handle it).

- It solves the **non-stopping** problem in many cases.
- The key insight is to **relax the condition of strict tangency** of the vector fields X to the primary and secondary constraint hypersurfaces $\mathcal{M}_k$.
- This is done in a way that only changes the algorithm for field theories but **not for mechanical systems**.
- As you can probably guess this change takes advantage of the **richer topological properties** of the infinite dimensional manifolds that are used for field theories.
- The steps of the algorithm are the same that had to be used before. The only change is in the definition of the $\mathcal{M}_k$ submanifolds.

Final description of the algorithm: It consists in the obtention of a family of (generalized) submanifolds $\mathcal{M}_k$ of the primary constraint submanifold $\mathcal{M}_1$, a family of maps (immersions) $\jmath_i$, $i \in \mathbb{N}$, and a vector field X.

- $\mathcal{M}_{k+1} \xrightarrow{\jmath_{k+1}} \mathcal{M}_k \xrightarrow{\jmath_k} \cdots \xrightarrow{\jmath_3} \mathcal{M}_2 \xrightarrow{\jmath_2} \mathcal{M}_1$ (primary)
- Each $\mathcal{M}_k$ is a Banach manifold modeled on a Banach space F_k.
- $\mathcal{M}_{k+1} := \{m \in \mathcal{M}_k \,:\, \mathrm{d}H(m) \in \flat(T\overline{\mathcal{M}}_k|_{\jmath_2 \circ \cdots \circ \jmath_k(\mathcal{M}_k)})\}$
- $\overline{\mathcal{M}}_k := \mathrm{cl}_{\mathcal{M}_1}(\jmath_2 \circ \cdots \circ \jmath_k(\mathcal{M}_k)) \subset \mathcal{M}_1$.
- The $\jmath_i : \mathcal{M}_i \to \mathcal{M}_{i-1}$ are smooth injective immersions.
- The smallest $n \in \mathbb{N}$ such that $\mathcal{M}_{n+1} = \mathcal{M}_n \neq \emptyset$ (if it exists) provides us with:
  1. A generalized submanifold $\mathcal{N}$ (topology not the induced one) of $\mathcal{M}_1$.
  2. A smooth inclusion $\jmath = \jmath_2 \circ \cdots \circ \jmath_n : \mathcal{N} \to \mathcal{M}_1$.
  3. vector fields X solving $(i_X\omega - \mathrm{d}H)|_{\mathcal{N}} = 0$.

1. **Scalar fields:**
   - Dirichlet B.C.: $\Phi = 0$ on $\partial\Sigma$.
   - Robin B.C.: $\vec{n} \cdot \vec{\nabla}\Phi = B\Phi$ on $\partial\Sigma$.
2. **Electromagnetic field:**
   - Perfect conductor B.C.: $\vec{n} \times \vec{A} = \vec{0}$, $A_\perp = 0$ on $\partial\Sigma$.
   - Neumann B.C.: $\vec{n} \cdot (\dot{\vec{A}} + \vec{\nabla} A_\perp) = 0$, $\vec{n} \times (\vec{\nabla} \times \vec{A})$ on $\partial\Sigma$.

Some notation and conventions:

- $\Sigma \subset \mathbb{R}^3$ open, connected, bounded with smooth boundary.
- $\langle \cdot, \cdot \rangle_{L^2}$ is the scalar product in $L^2(\Sigma)$.
- $H^1(\Sigma)$ Sobolev space $H^1(\Sigma) = \{u \in L^2(\Sigma) : u' \in L^2(\Sigma)\}$.
- $H^1(\Sigma)$ is a Hilbert space with the scalar product $\langle u_1, u_2 \rangle_{H^1} = \langle u_1, u_2 \rangle_{L^2} + \langle \vec{\nabla} u_1, \vec{\nabla} u_2 \rangle_{\vec{L}^2}$.
- Trace operator $\gamma : H^1(\Sigma) \to L^2(\partial\Sigma)$. Bounded operator that, restricted to continuous functions gives their boundary values. If $u \in H^1(\Sigma)$ I will denote $u|_{\partial\Sigma} := \gamma(u)$.

- $L_D : H_0^1(\Sigma) \times L^2(\Sigma) \to \mathbb{R} : (Q, V) \mapsto \frac{1}{2}\langle V, V\rangle_{L^2(\Sigma)} - \frac{1}{2}\langle \vec{\nabla} Q, \vec{\nabla} Q\rangle_{\vec{L}^2(\Sigma)}$
- Here $H_0^1(\Sigma) = \{u \in H^1(\Sigma) : u|_{\partial\Sigma} = 0\}$.
- $\mathcal{M}_1 = H_0^1(\Sigma) \times L^2(\Sigma) =: \mathcal{M}$, (primary constraint manifold).
- $\mathcal{M}_2 = \left(H^2(\Sigma) \cap H_0^1(\Sigma)\right) \times H_0^1(\Sigma) =: \mathcal{N}$, (secondary constraint manifold).
- $\jmath_2 : \mathcal{M}_2 \to \mathcal{M}_1$ inclusion map.
- $X : \mathcal{N} \to \mathcal{N} \times \mathcal{M} : (Q, P) \mapsto ((Q, P), (P, \Delta_D Q))$. The integral curves are given by
$$\begin{aligned} \dot{Q} &= P \\ \dot{P} &= \Delta_D Q \end{aligned}$$
- $\Delta_D : H^2(\Sigma) \cap H_0^1(\Sigma) \to L^2(\Sigma)$ is the scalar Dirichlet Laplacian.

Remark: If we had used the strict tangency condition we would have found the following (well known) infinite chain of constraints:

$$Q|_{\partial\Sigma} = 0\,; \;\; P|_{\partial\Sigma} = 0\,; \;\ldots, \Delta_D^k Q|_{\partial\Sigma} = 0\,; \;\; \Delta_D^k P|_{\partial\Sigma} = 0\,, \ldots$$

Scalar field (Robin B.C.)

- $L_R : H^1(\Sigma) \times L^2(\Sigma) \to \mathbb{R}$
  $L_R(Q,V) = \frac{1}{2}\langle V, V\rangle_{L^2} - \frac{1}{2}\langle \vec{\nabla} Q, \vec{\nabla} Q\rangle_{\vec{L}^2} + \int_{\partial\Sigma} \big(AQ|_{\partial\Sigma} + \frac{B}{2} Q^2|_{\partial\Sigma}\big)\mathrm{vol}_{\partial\Sigma}$
- Here $A, B \in C^\infty(\partial\Sigma)$ with $B \leq 0$. If $A = B = 0$ we have the Neumann Lagrangian.
- $\mathcal{M}_1 = H^1(\Sigma) \times L^2(\Sigma) =: \mathcal{M}$ (primary constraint manifold).
- $\mathcal{M}_2 = \big(H^2_\partial(\Sigma) \cap H^1(\Sigma)\big) \times H^1(\Sigma) =: \mathcal{N}$ (secondary constraint manifold).
- Notation $H^2_\partial(\Sigma) := \{Q \in H^2(\Sigma) : (BQ - \vec{n}\cdot\vec{\nabla} Q)|_{\partial\Sigma} = 0\}$.
- $\jmath_2 : \mathcal{M}_2 \to \mathcal{M}_1$ inclusion map.
- $X : \mathcal{N} \to \mathcal{N} \times \mathcal{M} : (Q,P) \mapsto ((Q,P),(P,\Delta_R Q))$. The integral curves are given by
  $$\begin{aligned} \dot{Q} &= P \\ \dot{P} &= \Delta_R Q \end{aligned}$$
- $\Delta_R : H^2(\Sigma) \cap H^1(\Sigma) \to L^2(\Sigma)$ is the scalar Robin Laplacian.

The EM field (perfect conductor B.C.)

- $L_D : \mathbf{H}^1_\partial(\Sigma) \times \mathbf{L}^2(\Sigma) \to \mathbb{R}$

$$L_D(Q,V) = \frac{1}{2}\langle \vec{V}, \vec{V}\rangle_{\vec{L}^2} + \langle \vec{\nabla} Q_\perp, \vec{V}\rangle_{\vec{L}^2} + \frac{1}{2}\langle \vec{\nabla} Q_\perp, \vec{\nabla} Q_\perp\rangle_{\vec{L}^2} - \frac{1}{2}\langle \vec{\nabla}\times\vec{Q}, \vec{\nabla}\times\vec{Q}\rangle_{\vec{L}^2}$$

- Here $\mathbf{L}^2 := L^2_\perp(\Sigma) \times \vec{L}^2(\Sigma)$, $\mathbf{H}^1_\partial(\Sigma) = H^1_{0\perp}(\Sigma) \times \vec{H}_0(\mathrm{curl}, \Sigma)$
- $\mathcal{M}_1 = \mathbf{H}^1_\partial(\Sigma) \times \mathbf{L}^2(\Sigma) =: \mathcal{M}$ (primary constraint manifold).
- $\mathcal{M}_2 = \{(Q, \vec{P}) \,:\, Q_\perp \in H^1_{0\perp}(\Sigma),\ \vec{Q} \in \vec{H}^2_\partial(\mathrm{curl}, \Sigma),$
  $\vec{P} \in \vec{H}_0(\mathrm{curl}, \Sigma) \cap \vec{H}(\mathrm{div}, \Sigma),\ \vec{\nabla}\cdot\vec{P} = 0\}$
  (secondary constraint manifold).
- $X : \mathcal{N} \to \mathcal{N}\times\mathcal{M} : (Q, \vec{P}) \mapsto ((Q, \vec{P}), (X_{Q_\perp}, \vec{P} - \vec{\nabla} Q_\perp, -\vec{\nabla}\times\vec{\nabla}\times\vec{Q}))$.
  The integral curves are given by

$$\begin{aligned} \dot{Q}_\perp &\in H^1_{0\perp} \\ \dot{Q} &= \vec{P} - \vec{\nabla} Q_\perp \\ \dot{P} &= -\vec{\nabla}\times\vec{\nabla}\times\vec{Q} \end{aligned}$$

- The generalized submanifold $\mathcal{M}_2 \xrightarrow{\jmath_2} \mathcal{M}_1$ is **first class**.

The EM field (Neumann B.C.)

- $L_N : \mathbf{H}^1(\Sigma) \times \mathbf{L}^2(\Sigma) \to \mathbb{R}$

$$L_N(Q,V) = \frac{1}{2}\langle \vec{V}, \vec{V}\rangle_{\vec{L}^2} + \langle \vec{\nabla} Q_\perp, \vec{V}\rangle_{\vec{L}^2} + \frac{1}{2}\langle \vec{\nabla} Q_\perp, \vec{\nabla} Q_\perp\rangle_{\vec{L}^2} - \frac{1}{2}\langle \vec{\nabla}\times\vec{Q}, \vec{\nabla}\times\vec{Q}\rangle_{\vec{L}^2}$$

- Now $\mathbf{H}^1(\Sigma) = H^1_\perp(\Sigma) \times \vec{H}(\mathrm{curl}, \Sigma)$
- $\mathcal{M}_1 = \mathbf{H}^1(\Sigma) \times \mathbf{L}^2(\Sigma) =: \mathcal{M}$ (primary constraint manifold).
- $\mathcal{M}_2 = \{(Q, \vec{P}) \in \mathbf{H}^2(\Sigma) \times (\vec{H}(\mathrm{curl}, \Sigma) \cap \vec{H}_0(\mathrm{div}, \Sigma))$
  $: \vec{\nabla}\cdot\vec{P} = 0\,, \quad \vec{n}\times(\vec{\nabla}\times\vec{Q})|_{\partial\Sigma} = \vec{0}\}$
  (secondary constraint manifold).
- $X : \mathcal{N} \to \mathcal{N}\times\mathcal{M} : (Q, \vec{P}) \mapsto ((Q, \vec{P}), (X_{Q_\perp}, \vec{P} - \vec{\nabla} Q_\perp, -\vec{\nabla}\times\vec{\nabla}\times\vec{Q}))$. The integral curves are given by

$$\begin{array}{rcl} \dot{Q}_\perp \in H^1_\perp & & \\ \dot{\vec{Q}} & = & \vec{P} - \vec{\nabla} Q_\perp \\ \dot{\vec{P}} & = & -\vec{\nabla}\times\vec{\nabla}\times\vec{Q} \end{array}$$

- Again, the generalized submanifold $\mathcal{M}_2 \xrightarrow{\jmath_2} \mathcal{M}_1$ is **first class**.

Conclusions and Comments:

- The Hamiltonian treatment of field theories requires some care...
- Functional analytic issues must be taken into account.
- This is especially true in the presence of boundaries.
- If you want to do Dirac quantization use Dirac's method (if possible).
- If you think that you can handle the reduced phase space then GNH is better.
- The GNH approach works in situations where the Dirac approach fails.
- It is the basis of the Lagrangian-symplectic approach.

Questions:

- Are there theories with genuine boundary degrees of freedom?
- The case of isolated horizons, Are there any extra boundary conditions similar to the ones appearing for the scalar field?